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Elastic equations for a cylindrical section of a tree

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Abstract

Considering a cylindrical section of a tree subjected to loads independent of x_3 as a relaxed Saint-Venant's problem, it was shown that plane sections remain plane. Since plane sections remain plane, the displacement equations for the neutral fiber derived using either the relaxed Saint-Venant's problem or elementary beam theory are equivalent. The stresses in the plane of the transverse cross-section were found to equal to zero. Therefore, it is appropriate to use elementary beam theory to estimate the three-dimensional stress functions when the wood is considered to be homogeneous. In addition the three-dimensional displacement equations allow the required elastic coefficients in cylindrical coordinates to be measured from full size samples.

Keywords: Elasticity; Anisotropy; Orthotropic; Wood; Cantilever beam

1. Introduction

This paper is the first of three papers that will consider the mechanical stresses in a cylindrical section of the bole of a tree. Depending on the analysis that is being performed different constitutive equations may be assumed for the wood in a tree. This paper will consider the wood to be homogeneous and orthotropic with respect to the cylindrical coordinates, with the z-axis directed up the tree. In the second and third papers the constitutive equations will depend on the radial coordinate r.

Fung (1965, p. 16) notes that problems such as wave propagation, oscillation, and contact problems may be "beyond the scope of the elementary theory" when considering anisotropic materials. Researchers in forestry and wood science who require estimates of the stresses on planes other than the transverse cross-section, need to know if elementary beam theory will provide adequate estimates of the stresses given an assumed constitutive equation. In addition, Pyles et al. (1988) noted that the elastic properties for minor specimens published by the American Society for Testing and Materials (ASTM) under estimated the

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stiffness of full size samples. Pyles et al. (1988) suggested that the preparation of the ASTM specimens might reduce the stiffness of the specimens.

Lyons (1997) searched for the maximum shear stress in the bole of a tree subject to combined loading, and found the maximum shear stress did not occur on the transverse cross-section. Elementary beam theory provides estimates of the normal stress in the axial direction (S_{33}), and the shear stresses (S_{13} and S_{23}) on the transverse cross-section. Consider the cylindrical section of a tree and the differential element shown in Fig. 1.

If the x_3-axis is directed up the tree then the normal vector on the transverse cross-section (Σ) is $\mathbf{n}^{(\Sigma)} = (0, 0, 1)$, and the stress vector on Σ is

$$\mathbf{s}^{(\Sigma)} = S_{ij}n_j^{(\Sigma)} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} S_{13} \\ S_{23} \\ S_{33} \end{bmatrix} \tag{1.1}$$

In the following Greek indices range from 1 to 2, while Latin indices range from 1 to 3 unless otherwise specified. Summation notation is used for repeated indices and a comma followed by a subscript will indicate a partial derivative with respect to the indicated coordinate. In addition, the following special functions will be used, the Kronecker delta function (δ_{ij}), and the two-dimensional alternator symbol ($e_{\alpha\beta}$).

Elementary beam theory will provide estimates for the stresses required in (1.1). However, if the stress vector is required on an arbitrary cross-section Ω with normal vector $\mathbf{n}^{(\Omega)} = n_j^{(\Omega)}$, then the stress vector on Ω is

$$\mathbf{s}^{(\Omega)} = S_{ij}n_j^{(\Omega)} \tag{1.2}$$

If $S_{\alpha\beta} = 0$, then (1.2) becomes

$$\mathbf{s}^{(\Omega)} = \begin{bmatrix} S_{13}n_1 \\ S_{23}n_2 \\ S_{3j}n_j \end{bmatrix} \tag{1.3}$$

Elementary beam theory will provide estimates for the stresses required in Eq. (1.3); however, if $S_{\alpha\beta} \neq 0$, then the stress vector on Ω becomes

$$\mathbf{s}^{(\Omega)} = \begin{bmatrix} S_{1j}n_j \\ S_{2j}n_j \\ S_{3j}n_j \end{bmatrix} \tag{1.4}$$

and elementary beam theory will not provide estimates for all the required stresses.

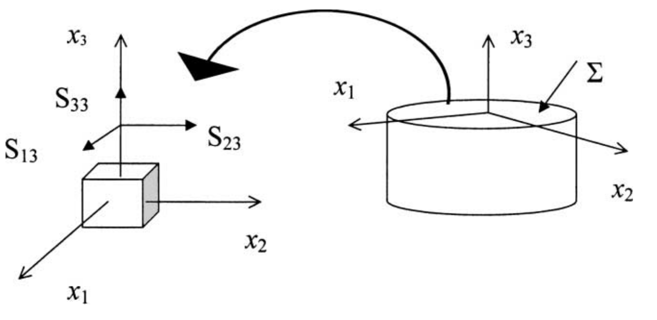

Fig. 1. Stresses acting on a transverse cross-section of a tree.

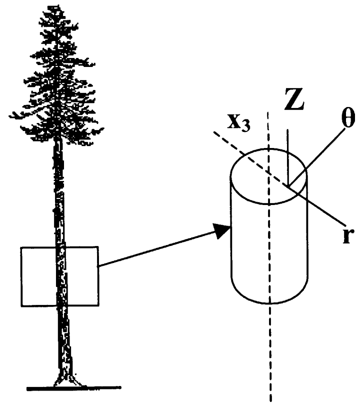

Fig. 2. Axes of symmetry in a cylindrical section of a tree.

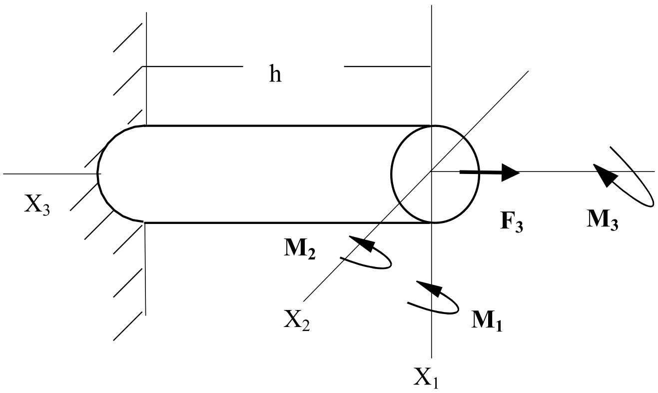

Fig. 3. Cylindrical cantilever beam subject to loads independent of x_3.

Consider a cylindrical section of a tree (Fig. 2). Bodig and Jayne (1993) describe a cylindrical section of a tree as being an orthotropic material with cylindrical anisotropy, where the axes of symmetry are the long axis z, the radial axis r, and the tangential axis θ. The problem considered in this paper is a cylindrical section of a tree that is fixed at one end and subject to loads independent of x_3 (Fig. 3). This paper has two objectives. First, elastic theory will be used to determine if elementary beam theory is appropriate for estimating the three-dimensional stresses in a cylindrical section of a tree, for loads independent of x_3. Second, the displacement equations will be derived for a cylindrical section of a tree so that the elastic coefficients in cylindrical coordinates may be measured on full size specimens.

2. Constitutive equations

The constitutive equations for a linear elastic material that is orthotropic in cylindrical coordinates are (prime denotes basis in cylindrical coordinates)

$$\begin{aligned} S'_{ij} &= C'_{ijkl}E'_{kl} \\ E'_{ij} &= S'_{ijkl}S'_{kl} \end{aligned} \tag{2.1}$$

where S'_{ij} is Cauchy's stress tensor, E'_{ij} is the infinitesimal strain tensor, C'_{ijkl} is the elasticity tensor, and S'_{ijkl} is the compliance tensor.

Note, in (2.1) the following constants must be equal to zero for an orthotropic material:

$$C'_{1123} = C'_{1113} = C'_{1112} = C'_{2223} = C'_{2213} = C'_{2212} = 0$$
$$C'_{3323} = C'_{3313} = C'_{3312} = C'_{2313} = C'_{2312} = C'_{1312} = 0$$
$$(\text{similarly for } S'_{ijkl})$$

Eq. (2.1) are tensor equations and so are valid under any proper transformation; however, it will be necessary to take the derivatives of these equations. If Eq. (2.1) has a curvilinear basis, then on taking the derivative with respect to a base vector the resulting differential will have a different set of base vectors from the point where the derivative was taken (Charlier et al., 1992, p. 21). The resulting matrix is no longer a tensor, and will have to be corrected in order to regain the original properties of the tensor equation. This complication can be avoided if the constitutive equations are transformed to a rectilinear basis. Then the base vectors are the same for all points in the domain and so taking the derivative of a tensor will result in a tensor.

Lai et al. (1993, p. 221) give the transformation taking the fourth order tensor C'_{ijkl} from the $\mathbf{e}'_\mathbf{i}$ basis to the $\mathbf{e}'_\mathbf{i}$ basis as,

$$\begin{aligned} C_{ijkl} &= Q_{mi}Q_{nj}Q_{rk}Q_{sl}C'_{mnrs} \\ S_{ijkl} &= Q_{mi}Q_{nj}Q_{rk}Q_{sl}S'_{mnrs} \end{aligned} \tag{2.2a}$$

Given (2.2a) the constitutive equations can be written in Cartesian coordinates

$$S_{ij} = C_{ijkl}E_{kl}, \quad E_{ij} = S_{ijkl}S_{kl} \tag{2.2b}$$

Here, Q_{ij} is the second order tensor containing the direction cosines for the rotation of interest. To convert the elasticity tensor or the compliance tensor from a cylindrical basis to a Cartesian basis Q_{ij} would be

$$Q_{ij} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.3}$$

where θ is the cylindrical coordinate.

The rotation (2.3) takes the positive r-direction in cylindrical coordinates to the positive x_1-direction in Cartesian coordinates. For the complete list of transformation equations resulting from (2.2a), refer to Appendix A. Recall for an orthotropic material there are only nine independent coefficients in the C'_{ijkl} and S'_{ijkl} tensors. Appendix A shows that there are now 13 nonzero coefficients after transforming the C'_{ijkl} and S'_{ijkl} tensors to Cartesian coordinates. In addition, the coefficients in the new C_{ijkl} and S_{ijkl} tensors are no longer constant; instead, they are now dependent on the cylindrical coordinate θ.

Let a cylindrical section of a tree be solid and orthotropic in cylindrical coordinates with constant coefficients, and let the x_3-axis be an axis of symmetry that falls within the body. The cylindrical base vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are not unique at $r = 0$, therefore, the constitutive equations must allow for nonunique strains in these directions at $r = 0$. Lekhnitskii (1981, p. 69) notes if the compliance and elasticity coefficients are constant then certain terms within the elasticity tensor or the compliance tensor must be equal.

The terms that must be equal are

$$\begin{aligned} &S'_{1111} = S'_{2222}, \quad S'_{1133} = S'_{2233}, \quad S'_{2332} = S'_{1313} \\ &C'_{1111} = C'_{2222}, \quad C'_{1133} = C'_{2233}, \quad C'_{2332} = C'_{1313} \end{aligned} \tag{2.4}$$

Eq. (2.4) reduces the number of independent coefficients in (2.1) from nine to six.

To view the change in dependence between the stresses and strains after the transformation (2.2a) the constitutive equations can be written in Voigt notation when taking (2.4) and (A.1) into account, for example

$$\begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{23} \\ S_{13} \\ S_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & C_{1112} \\ C_{1122} & C_{2222} & C_{2233} & 0 & 0 & C_{2212} \\ C_{1133} & C_{2233} & C_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{1313} & 0 \\ C_{1112} & C_{2212} & 0 & 0 & 0 & C_{1212} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{13} \\ 2E_{12} \end{bmatrix} \tag{2.5}$$

The coefficients in (2.5) have the following dependence

$$\begin{aligned} &C_{1111} = C_{2222} = C_{1111}(\theta), \quad C_{1122} = C'_{1122} \\ &C_{1133} = C_{2233} = C'_{1133}, \quad C_{3333} = C'_{3333} \\ &C_{2323} = C_{1313} = C'_{2323}, \quad C_{1212} = C_{1212}(\theta) \\ &C_{1112} = C_{1112}(\theta), \quad C_{2212} = C_{2212}(\theta) \end{aligned} \tag{2.6}$$

3. Problem statement

Iesan (1987) formulates a solution for a cylindrical cantilever beam with anisotropy that is dependent on the x_1- and x_2-coordinates. When the elasticity and compliance tensors in (2.1) are transformed into the Cartesian frame, they become functions of the x_1- and x_2-coordinates. Therefore, Iesan's solution may be used for the problem of a cylindrical section of a tree considered as a relaxed Saint-Venant's problem. Chirita (1979) uses Iesan's results to formulate the stress and displacement equations for a cylindrical cantilever beam made of a material with constant coefficients in Cartesian coordinates. As will be seen the simplifications resulting from (2.4) will allow a solution very similar to Chirita's.

Consider a cylindrical section of a tree as a cantilever beam with constant cross-sections (Fig. 3). Let Σ_1 be the open cross-section at $x_3 = 0$, let Σ_2 be the open cross-section at $x_3 = h$, and let Σ be an arbitrary cross-section with normal x_3. The lateral surface of the cylinder will be Π, while the boundary of an arbitrary cross-section is Γ.

The resultant loads applied to the cross-section at $x_3 = 0$ are the forces **F** and the moments **M**, the lateral surface is unloaded, the cross-section at Σ_2 is fixed, and body loads will be ignored in this analysis. The problem in Fig. 3 is of the class P_1 as defined by Iesan (1987), where the resultant loads acting on Σ are independent of x_3 and $F_\alpha = 0$.

Recall from (2.2a) that

$$C_{ijkl} = C_{ijkl}(x_1, x_2), \quad \text{and} \quad S_{ijkl} = S_{ijkl}(x_1, x_2) \tag{3.1}$$

The total displacements are

$$u_i^0 = u_i + u_i^I \tag{3.2}$$

where u_i are the displacements resulting from strain, and u_i^I are displacements resulting from a rigid body motion.

The displacements resulting from strain, derived in a manner similar to that used by Iesan (1987), are

$$u_i = \delta_{i\alpha}\left[-a_\alpha \frac{x_3^2}{2} + e_{\beta\alpha} a_4 x_\beta x_3 \right] + \delta_{i3}[a_\rho x_\rho + a_3]x_3 + W_i \tag{3.3}$$

where a_p are constants that will have to be determined using the boundary conditions, and $\mathbf{W} = \mathbf{W}(x_1, x_2)$ is a vector composed of the functions of integration.

The displacements resulting from a rigid body motion are

$$\begin{aligned} u_1^I &= -w_3x_2 + w_2x_3 + u_{10} \\ u_2^I &= w_3x_1 - w_1x_3 + u_{20} \\ u_3^I &= w_1x_2 - w_2x_1 + u_{30} \end{aligned} \tag{3.4}$$

where w_i are rotations about the x_i-axes, and u_{i0} are translations in the x_i-directions.

Since the body forces are being ignored and the lateral surface of the cylinder is unloaded, the necessary conditions for a solution imply that the sum of the stress fields acting on Σ_2 must be in equilibrium with the resultant loads acting on Σ_1

$$\begin{aligned} &\int_{\Sigma_2} S_{\alpha 3}(\mathbf{u})\,\mathrm{d}a = -f_\alpha(\mathbf{u}) = 0, \quad \int_{\Sigma_2} S_{33}(\mathbf{u})\,\mathrm{d}a = -f_3(\mathbf{u}) = -F_3 \\ &\int_{\Sigma_2} e_{\alpha\beta}x_\alpha S_{3\beta}(\mathbf{u})\,\mathrm{d}a = -m_3(\mathbf{u}) = -M_3, \quad \int_{\Sigma_2} x_\alpha S_{33}(\mathbf{u})\,\mathrm{d}a = e_{\alpha\beta}m_\beta(\mathbf{u}) = e_{\alpha\beta}M_\beta \end{aligned} \tag{3.5}$$

Substituting (3.3) into the definition of the infinitesimal strain tensor, the resulting strains are

$$\begin{aligned} &E_{11}(\mathbf{u}) = W_{1,1}, \quad E_{22}(\mathbf{u}) = W_{2,2}, \quad E_{33}(\mathbf{u}) = (a_\rho x_\rho + a_3) \\ &E_{23}(\mathbf{u}) = \tfrac{1}{2}[a_4x_1 + W_{3,2}], \quad E_{13}(\mathbf{u}) = \tfrac{1}{2}[-a_4x_2 + W_{3,1}], \quad E_{12}(\mathbf{u}) = \tfrac{1}{2}[W_{1,2} + W_{2,1}] \end{aligned} \tag{3.6}$$

Consider the constitutive equations (2.2b). Substitute the strain tensor (3.6) into the constitutive equations, then the stress tensor in Cartesian coordinates becomes

$$S_{ij}(\mathbf{u}) = C_{ij33}(a_\rho x_\rho + a_3) - a_4C_{ij\alpha3}e_{\alpha\beta}x_\beta + T_{ij}(\mathbf{W}) \tag{3.7}$$

The $T_{ij}(\mathbf{W}) = C_{ijk\alpha}W_{k,\alpha}$ are the stresses resulting from the displacement vector $\mathbf{W}$, which is independent of x_3 and so forms a generalized plain strain problem. Iesan (1987) found that the generalized plane strain problem could be separated into four auxiliary problems $T_{ij}^{(p)}$ $(p = 1, 2, 3, 4)$, which are defined by the following equilibrium equations and boundary conditions:

$$\begin{aligned} &T_{i\alpha}^{(\beta)}(\mathbf{W})_{,\alpha} + (C_{i\alpha33}x_\beta)_{,\alpha} = 0, \quad T_{i\alpha}^{(\beta)}(\mathbf{W})n_\alpha = -C_{i\alpha33}x_\beta n_\alpha \\ &T_{i\alpha}^{(3)}(\mathbf{W})_{,\alpha} + (C_{i\alpha33})_{,\alpha} = 0, \quad T_{i\alpha}^{(3)}(\mathbf{W})n_\alpha = -C_{i\alpha33}n_\alpha \\ &T_{i\alpha}^{(4)}(\mathbf{W})_{,\alpha} - e_{\rho\beta}(C_{i\alpha\rho3}x_\beta)_{,\alpha} = 0, \quad T_{i\alpha}^{(4)}(\mathbf{W})n_\alpha = e_{\rho\beta}C_{i\alpha\rho3}x_\beta n_\alpha \end{aligned} \tag{3.8}$$

Here $\mathbf{n}$ is the unit normal to Γ. The auxiliary problems combine as follows:

$$T_{ij}(\mathbf{W}) = \sum_{p=1}^{4} a_p T_{ij}^{(p)}(\mathbf{W}) \tag{3.9}$$

After substituting the stresses (3.7) into the necessary conditions for a solution (3.5), and taking note of the simplifications resulting from (2.5), the following system of equations can be found for determining a_p:

$$\begin{bmatrix} \int_{\Sigma_2} x_1^2 C_{3333}\,\mathrm{d}a & \int_{\Sigma_2} x_1x_2 C_{3333}\,\mathrm{d}a & \int_{\Sigma_2} x_1 C_{3333}\,\mathrm{d}a & 0 \\ \int_{\Sigma_2} x_1x_2 C_{3333}\,\mathrm{d}a & \int_{\Sigma_2} x_2^2 C_{3333}\,\mathrm{d}a & \int_{\Sigma_2} x_2 C_{3333}\,\mathrm{d}a & 0 \\ \int_{\Sigma_2} x_1 C_{3333}\,\mathrm{d}a & \int_{\Sigma_2} x_2 C_{3333}\,\mathrm{d}a & \int_{\Sigma_2} C_{3333}\,\mathrm{d}a & 0 \\ 0 & 0 & 0 & \int_{\Sigma_2} [x_1^2 C_{2323} + x_2^2 C_{1313}]\,\mathrm{d}a \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix} \tag{3.10}$$

where

$$\begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix} = \begin{bmatrix} M_1 - \int_{\Sigma_2} x_1 T_{33}\,\mathrm{d}a \\ M_2 - \int_{\Sigma_2} x_2 T_{33}\,\mathrm{d}a \\ -F_3 - \int_{\Sigma_2} T_{33}\,\mathrm{d}a \\ -M_3 - \int_{\Sigma_2} x_1 T_{32} + x_2 T_{31}\,\mathrm{d}a \end{bmatrix}$$

Recall from (2.6) that $C_{2323} = C_{1313} = C'_{2323}$ and $C_{3333} = C'_{3333}$, and that the integrals are taken over a circular cross-section. Therefore, since C'_{3333} and C'_{2323} are constant, Eq. (3.10) becomes

$$\begin{bmatrix} C'_{3333}I & 0 & 0 & 0 \\ 0 & C'_{3333}I & 0 & 0 \\ 0 & 0 & C'_{3333}A & 0 \\ 0 & 0 & 0 & 2C'_{2323}I \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix} \tag{3.11}$$

Here I is the moment of inertia, and A is the cross-sectional area. Since C'_{3333}, C'_{2323}, I, and A are never zero, then (3.11) uniquely defines a_p.

4. Generalized plane strain stresses T_{ij}

Recall for generalized plane strain that $E_{ij} = S_{ijmn}T_{mn}$. Chirita (1979) notes this process must be reversible, therefore, $T_{kl} = C_{klrs}E_{rs}$. This results in $E_{ij} = S_{ijmn}C_{mnrs}E_{rs}$, and

$$S_{ijmn}C_{mnrs} = \frac{1}{2}[\delta_{ir}\delta_{js} + \delta_{ij}\delta_{jr}] = \begin{cases} (i = s, j = r) \Rightarrow 1/2 \\ (i = r, j = s) \Rightarrow 1/2 \\ (i = r, j = s; i = s, j = r) \Rightarrow 1 \\ \text{all other } i, j, r, s \Rightarrow 0 \end{cases} \tag{4.1}$$

For the auxiliary generalized plane strain problems, the stresses and strains are functions of the displacement vector $\mathbf{W}(x_1, x_2)$

$$\begin{aligned} T^{(p)}_{ij}(\mathbf{W}) &= C_{ijkl}E^{(p)}_{kl}(\mathbf{W}) \\ E^{(p)}_{ij}(\mathbf{W}) &= S_{ijkl}T^{(p)}_{kl}(\mathbf{W}) \end{aligned} \tag{4.2}$$

Since $\mathbf{W}$ is independent of x_3, there is the following constraint on $E^{(p)}_{33}(\mathbf{W})$

$$E^{(p)}_{33}(\mathbf{W}) = \frac{1}{2}\left[\frac{\partial W^{(p)}_3}{\partial x_3} + \frac{\partial W^{(p)}_3}{\partial x_3}\right] = 0 \tag{4.3}$$

4.1. Functions for $T^{(1)}_{ij}$

Set $p = 1$ in (3.7) and (3.8), then the system of equations that defines $T^{(1)}_{ij}(\mathbf{W})$ is

$$T^{(1)}_{i\alpha}(\mathbf{W})_{,\alpha} + (C_{i\alpha 33}x_1)_{,\alpha} = 0 \tag{4.4}$$

$$T^{(1)}_{i\alpha}(\mathbf{W})n_\alpha = -C_{i\alpha 33}x_1 n_\alpha \tag{4.5}$$

Recall from (2.5) that $C_{1233} = 0$, therefore, let

$$T^{(1)}_{11} = -x_1 C_{1133}, \quad T^{(1)}_{22} = -x_1 C_{2233}, \quad T^{(1)}_{12} = -x_1 C_{1233} = 0, \quad T^{(1)}_{13} = T^{(1)}_{23} = 0 \tag{4.6}$$

It can be seen that the stress functions (4.6) satisfy (4.4) and (4.5). To obtain a function for $T_{33}^{(1)}(\mathbf{W})$ expand (4.2) with $i = j = 3$, then

$$E_{33}^{(1)} = S_{3311}T_{11}^{(1)} + S_{3322}T_{22}^{(1)} + S_{3333}T_{33}^{(1)} + 2S_{3312}T_{12}^{(1)} = 0 \tag{4.7}$$

Solve (4.7) for $T_{33}^{(1)}$ and then substitute (4.6) into this,

$$T_{33}^{(1)} = \frac{x_1}{S_{3333}}[S_{3311}C_{3311} + S_{3322}C_{3322}] \tag{4.8}$$

Recall from (4.1) that

$$S_{3311}C_{1133} + S_{3322}C_{2233} = 1 - S_{3333}C_{3333} \tag{4.9}$$

Therefore,

$$T_{33}^{(1)} = \frac{x_1}{S_{3333}}[1 - S_{3333}C_{3333}] = \frac{x_1}{S_{3333}} - x_1C_{3333} \tag{4.10}$$

The stress functions for the problems $T_{ij}^{(2)}(\mathbf{W})$ and $T_{ij}^{(3)}(\mathbf{W})$ can be found in a similar manner as for $T_{ij}^{(1)}(\mathbf{W})$.

4.2. Functions for $T_{ij}^{(4)}$

Set $p = 4$ in (3.7) and (3.8), then the system of equations that defines $T_{ij}^{(4)}(\mathbf{W})$ is

$$T_{i\alpha}^{(4)}(\mathbf{W})_{,\alpha} - e_{\rho\beta}(C_{i\alpha\rho3}x_\beta)_{,\alpha} = 0 \tag{4.11}$$

$$T_{i\alpha}^{(4)}(\mathbf{W})n_\alpha = e_{\rho\beta}C_{i\alpha\rho3}x_\beta n_\alpha \tag{4.12}$$

Recall from (2.5) that $C_{\gamma\alpha\rho3} = 0$, and from (2.6) that $C_{1313} = C_{2323} = C'_{2323}$. Let

$$T_{11}^{(4)} = T_{22}^{(4)} = T_{12}^{(4)} = T_{13}^{(4)} = T_{23}^{(4)} = 0 \tag{4.13}$$

It is easily seen that (4.13) satisfies all the equations of (4.11) and the first two equations of (4.12). Expanding the third equation of (4.12) and substituting in (4.13) results in

$$\begin{aligned}
&T_{31}^{(4)}n_1 + T_{32}^{(4)}n_2 = C_{3113}x_2n_1 - C_{3223}x_1n_2\\
&C'_{2323}x_2n_1 = C'_{2323}x_1n_2\\
&x_2n_1 = x_1n_2\\
&r\sin(\theta)\cos(\theta) = r\cos(\theta)\sin(\theta)
\end{aligned}$$

Therefore, (4.13) also satisfies the third equation of (4.12). To obtain a function for $T_{33}^{(4)}(\mathbf{W})$ expand (4.2) with $i = j = 3$ and substitute this into (4.3), then

$$E_{33}^{(4)} = S_{3311}T_{11}^{(4)} + S_{3322}T_{22}^{(4)} + S_{3333}T_{33}^{(4)} + 2S_{3312}T_{12}^{(4)} = 0 \tag{4.14}$$

Solve (4.14) for $T_{33}^{(4)}$ and substitute (4.13) into this, then

$$T_{33}^{(4)} = 0 \tag{4.15}$$

4.3. Summarizing the generalized plane strain stresses $T_{ij}^{(p)}$

The stresses that are a function of $\mathbf{W}^{(1)}$ are defined in Section 4.1. The stresses that are a function of $\mathbf{W}^{(2)}$ or $\mathbf{W}^{(3)}$ can be defined following methods similar to those shown in Section 4.1. The stresses that are a function of $\mathbf{W}^{(4)}$ were defined in Section 4.2. The four systems of stresses corresponding to the four displacement vectors $\mathbf{W}^{(p)}$ are as follows.

Consider $\mathbf{W}^{(1)}$; the corresponding stresses defined by (4.6) and (4.10) are

$$\begin{aligned} T_{11}^{(1)} &= -x_1 C_{1133}, \quad T_{22}^{(1)} = -x_1 C_{2233} \\ T_{12}^{(1)} &= -x_1 C_{1233}, \quad T_{23}^{(1)} = T_{13}^{(1)} = 0 \\ T_{33}^{(1)} &= x_1 S_{3333}^{-1} - x_1 C_{3333} \end{aligned} \tag{4.16}$$

Consider $\mathbf{W}^{(2)}$; the corresponding stresses derived similarly as for $\mathbf{W}^{(1)}$ are

$$\begin{aligned} T_{11}^{(2)} &= -x_2 C_{1133}, \quad T_{22}^{(2)} = -x_2 C_{2233} \\ T_{12}^{(2)} &= -x_2 C_{1233}, \quad T_{23}^{(2)} = T_{13}^{(2)} = 0 \\ T_{33}^{(2)} &= x_2 S_{3333}^{-1} - x_2 C_{3333} \end{aligned} \tag{4.17}$$

Consider $\mathbf{W}^{(3)}$; the corresponding stresses derived similarly as for $\mathbf{W}^{(1)}$ are

$$\begin{aligned} T_{11}^{(3)} &= -C_{1133}, \quad T_{22}^{(3)} = -C_{2233} \\ T_{12}^{(3)} &= -C_{1233}, \quad T_{23}^{(3)} = T_{13}^{(3)} = 0 \\ T_{33}^{(3)} &= S_{3333}^{-1} - C_{3333} \end{aligned} \tag{4.18}$$

Consider $\mathbf{W}^{(4)}$; the corresponding stresses from (4.13) and (4.15) are

$$T_{11}^{(4)} = T_{22}^{(4)} = T_{12}^{(4)} = T_{33}^{(4)} = T_{13}^{(4)} = T_{23}^{(4)} = 0 \tag{4.19}$$

5. Forming the total stresses S_{ij} and determining the constants a_p

To form the total stresses $S_{ij}(\mathbf{u})$, it will be necessary to combine the auxiliary generalized plain strain problems. Recall Eq. (3.9)

$$T_{ij}(\mathbf{W}) = \sum_{p=1}^{4} a_p T_{ij}^{(p)}(\mathbf{W}) \tag{3.9}$$

Substitute (4.16)–(4.19) into (3.9), then

$$T_{ij}(\mathbf{W}) = (a_\rho x_\rho + a_3)[-C_{ij33} + \delta_{i3}\delta_{j3} S_{3333}^{-1}] \tag{5.1}$$

Substitute (5.1) into (3.7) and cancel terms, then the stress tensor becomes

$$S_{ij}(\mathbf{u}) = -a_4 C_{ij\alpha 3} e_{\alpha\beta} x_\beta + (a_\rho x_\rho + a_3)\delta_{i3}\delta_{j3} S_{3333}^{-1} \tag{5.2}$$

The coefficients in (5.2) can be determined by substituting (5.2) into the necessary conditions for a solution (3.5). The first two equations in (3.5) are identically satisfied by (5.2) for all a_4.

Substitute (5.2) into the last four equations of (3.5), then

$$\begin{aligned}
&\int_{\Sigma_2} [a_\rho x_\rho + a_3] S_{3333}^{-1}\,\mathrm{d}a = -F_3 \\
&\int_{\Sigma_2} [a_4 C_{3223} x_1^2 + a_4 C_{3113} x_2^2]\,\mathrm{d}a = -M_3 \\
&\int_{\Sigma_2} x_1 [a_\rho x_\rho + a_3] S_{3333}^{-1}\,\mathrm{d}a = M_2 \\
&\int_{\Sigma_2} x_2 [a_\rho x_\rho + a_3] S_{3333}^{-1}\,\mathrm{d}a = -M_1
\end{aligned} \tag{5.3}$$

Recall that Σ_2 is a circular cross-section and that $C_{1313} = C_{2323} = C'_{2323}$ and $S_{3333} = S'_{3333}$, therefore, solving (5.3) for a_p results in

$$\begin{aligned}
&a_1 = \frac{4}{\pi R^4} M_2 S'_{3333}, \quad a_2 = \frac{-4}{\pi R^4} M_1 S'_{3333} \\
&a_3 = \frac{-F_3 S'_{3333}}{\pi R^2}, \quad a_4 = \frac{-2M_3}{C'_{2323} \pi R^4}
\end{aligned} \tag{5.4}$$

where R is the radius of the cross-section.

Substitute (5.4) into (5.2) then the stress tensor becomes

$$\begin{aligned}
&S_{11}(\mathbf{u}) = 0, \quad S_{22}(\mathbf{u}) = 0 \\
&S_{12}(\mathbf{u}) = 0, \quad S_{13}(\mathbf{u}) = \frac{2x_2 M_3}{\pi R^4} \\
&S_{23}(\mathbf{u}) = \frac{-2x_1 M_3}{\pi R^4}, \quad S_{33}(\mathbf{u}) = \frac{4x_1 M_2}{\pi R^4} - \frac{4x_2 M_1}{\pi R^4} - \frac{F_3}{\pi R^2}
\end{aligned} \tag{5.5}$$

6. Forming the displacement equations resulting from strain

Substitute (2.2b) into (3.6) and recall that $\mathbf{W} = \mathbf{W}(x_1, x_2)$, then

$$\begin{aligned}
&E_{11}(\mathbf{u}) = \frac{\partial u_1}{\partial x_1} = W_{1,1} = S_{11kl} S_{kl} \\
&E_{22}(\mathbf{u}) = \frac{\partial u_2}{\partial x_2} = W_{2,2} = S_{22kl} S_{kl}
\end{aligned} \tag{6.1}$$

Substitute (5.5) into (6.1) and integrate to find W_1 and W_2

$$\begin{aligned}
&W_1(x_1, x_2) = S_{1133}\left[\frac{2x_1^2 M_2}{\pi R^4} - \frac{4x_2 x_1 M_1}{\pi R^4} - \frac{x_1 F_3}{\pi R^2}\right] + K_1(x_2) \\
&W_2(x_1, x_2) = S_{2233}\left[\frac{4x_2 x_1 M_2}{\pi R^4} - \frac{2x_2^2 M_1}{\pi R^4} - \frac{x_2 F_3}{\pi R^2}\right] + K_2(x_1)
\end{aligned} \tag{6.2}$$

Substitute (6.2) into (3.3) for $i = 1, 2$

$$\begin{aligned}
&u_1 = -a_1 \frac{x_3^2}{2} - a_4 x_2 x_3 + S_{1133}\left[\frac{2x_1^2 M_2}{\pi R^4} - \frac{4x_2 x_1 M_1}{\pi R^4} - \frac{x_1 F_3}{\pi R^2}\right] + K_1(x_2) \\
&u_2 = -a_2 \frac{x_3^2}{2} + a_4 x_1 x_3 + S_{2233}\left[\frac{4x_2 x_1 M_2}{\pi R^4} - \frac{2x_2^2 M_1}{\pi R^4} - \frac{x_2 F_3}{\pi R^2}\right] + K_2(x_1)
\end{aligned} \tag{6.3}$$

Recall the definition of $E_{12}(\mathbf{u})$ from (3.6) and substitute (6.2) into this

$$E_{12}(\mathbf{u}) = \frac{1}{2}\left[\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right] = \frac{1}{2}\left[-a_4x_3 + S_{1133}\frac{4x_1M_1}{\pi R^4} + K_{1,2}\right] + \frac{1}{2}\left[a_4x_3 + S_{2233}\frac{4x_2M_2}{\pi R^4} + K_{2,1}\right] \tag{6.4}$$

From (2.2b) $E_{12}(\mathbf{u}) = S_{12kl}S_{kl}$, however, from (2.5) and (5.5) $S_{12kl}S_{kl} = 0$. Therefore, (6.4) becomes

$$K_{1,2} + K_{2,1} = \left[S_{1133}\frac{4x_1M_1}{\pi R^4}\right] - \left[S_{2233}\frac{4x_2M_2}{\pi R^4}\right] \tag{6.5}$$

Since K_1 is only a function of x_2 and K_2 is only a function of x_1, then from (6.5) K_1 and K_2 must be

$$\begin{aligned} K_1 &= -S_{2233}\frac{2x_2^2M_2}{\pi R^4} + L_1 \\ K_2 &= S_{1133}\frac{2x_1^2M_1}{\pi R^4} + L_2 \end{aligned} \tag{6.6}$$

where L_1 and L_2 are constants.

Recall from (4.16)–(4.19) that $T_{23}^{(p)} = T_{13}^{(p)} = 0$, also recall (2.2b) and (3.9), then

$$\begin{aligned} E_{23}(\mathbf{W}) &= 2S_{2323}T_{23} + 2S_{2313}T_{13} \\ E_{23}(\mathbf{W}) &= 2S_{2323}\sum_{p=1}^{4} a_pT_{23}^{(p)} + 2S_{2313}\sum_{p=1}^{4} a_pT_{13}^{(p)} = 0 \end{aligned} \tag{6.7}$$

The W_3 component of the two-dimensional vector $\mathbf{W}(x_1, x_2)$ can be found by equating the definition of the infinitesimal strains to (6.7)

$$\begin{aligned} E_{23}(\mathbf{W}) &= \tfrac{1}{2}[W_{2,3} + W_{3,2}] = \tfrac{1}{2}W_{3,2} = 0 \\ E_{13}(\mathbf{W}) &= \tfrac{1}{2}[W_{1,3} + W_{3,1}] = \tfrac{1}{2}W_{3,1} = 0 \end{aligned} \tag{6.8}$$

Integrating both equations of (6.8) results in

$$\begin{aligned} W_3 &= K_3(x_1) \\ W_3 &= K_4(x_2) \end{aligned} \tag{6.9}$$

Therefore, $K_3(x_1) = K_4(x_2) = L_3$ and L_3 is a constant.

The three-dimensional displacement equations resulting from strain can be found by substituting (6.6) into (6.3) and (6.9) into (3.3) with $i = 3$. Recall (2.6) and the definitions of a_p from (5.4), then

$$\begin{aligned} u_1 &= -\frac{2x_3^2}{\pi R^4}M_2S'_{3333} + \frac{2M_3S'_{2323}}{\pi R^4}x_2x_3 + S'_{1133}\left[\frac{2x_1^2M_2}{\pi R^4} - \frac{4x_2x_1M_1}{\pi R^4} - \frac{x_1F_3}{\pi R^2} - \frac{2x_2^2M_2}{\pi R^4}\right] + L_1 \\ u_2 &= \frac{2x_3^2}{\pi R^4}M_1S'_{3333} - \frac{2M_3S'_{2332}}{\pi R^4}x_1x_3 + S'_{1133}\left[\frac{4x_2x_1M_2}{\pi R^4} - \frac{2x_2^2M_1}{\pi R^4} - \frac{x_2F_3}{\pi R^2} + \frac{2x_1^2M_1}{\pi R^4}\right] + L_2 \\ u_3 &= S'_{3333}\left[\frac{4x_1M_2}{\pi R^4} - \frac{4x_2M_1}{\pi R^4} - \frac{F_3}{\pi R^2}\right]x_3 + L_3 \end{aligned} \tag{6.10}$$

7. Forming the total displacement equations

Recall the total displacements are given by (3.2)

$$u_i^0 = u_i + u_i^I \tag{3.2}$$

Substitute (6.10) into (3.2)

$$
\begin{aligned}
u_1^0 &= -\frac{2x_3^2}{\pi R^4}M_2S'_{3333} + \frac{2M_3S'_{2323}}{\pi R^4}x_2x_3 + S'_{1133}\left[\frac{2[x_1^2 - x_2^2]M_2}{\pi R^4} - \frac{4x_2x_1M_1}{\pi R^4} - \frac{x_1F_3}{\pi R^2}\right] - w_3x_2 + w_2x_3 + u_{10} \\
u_2^0 &= \frac{2x_3^2}{\pi R^4}M_1S'_{3333} - \frac{2M_3S'_{2332}}{\pi R^4}x_1x_3 + S'_{1133}\left[\frac{4x_2x_1M_2}{\pi R^4} + \frac{2[x_1^2 - x_2^2]M_1}{\pi R^4} - \frac{x_2F_3}{\pi R^2}\right] + w_3x_1 - w_1x_3 + u_{20} \\
u_3^0 &= S'_{3333}\left[\frac{4x_1M_2}{\pi R^4} - \frac{4x_2M_1}{\pi R^4} - \frac{F_3}{\pi R^2}\right]x_3 + w_1x_2 - w_2x_1 + u_{30}
\end{aligned}
\tag{7.1}
$$

where $u_{i0} = u_{i0} + L_i$.

Recall from Fig. 3 that Σ_2 is fixed; this places the following constraints on the total displacements at $\mathbf{x} = (0, 0, h)$

$$
u_i^0 = \frac{\partial u_\alpha^0}{\partial x_3} = \frac{\partial u_2^0}{\partial x_1} = 0 \tag{7.2}
$$

Substitute (7.1) into the first two equations of (7.2) for $i = \alpha = 1$ at $\mathbf{x} = (0, 0, h)$ and solve for w_2 and u_{10}, then

$$
\begin{aligned}
w_2 &= \frac{4h}{\pi R^4}M_2S'_{3333} \\
u_{10} &= \frac{-2h^2}{\pi R^4}M_2S'_{3333}
\end{aligned}
\tag{7.3}
$$

Substitute (7.1) into the first two equations of (7.2) for $i = \alpha = 2$ at $\mathbf{x} = (0, 0, h)$ and solve for w_1 and u_{20}, then

$$
\begin{aligned}
w_1 &= \frac{-4h}{\pi R^4}M_1S'_{3333} \\
u_{20} &= \frac{-2h^2}{\pi R^4}M_1S'_{3333}
\end{aligned}
\tag{7.4}
$$

Substitute (7.1) into the first equation of (7.2) for $i = 3$ and the fourth equation of (7.2) at $\mathbf{x} = (0, 0, h)$ and solve for w_3 and u_{30}, then

$$
\begin{aligned}
w_3 &= \frac{2h}{\pi R^4}M_3S'_{2323} \\
u_{30} &= \frac{h}{\pi R^2}F_3S'_{3333}
\end{aligned}
\tag{7.5}
$$

To form the total displacement equations substitute (7.3)–(7.5) into (7.1), then

$$
\begin{aligned}
u_1^0 &= -\frac{M_2S'_{3333}}{\pi R^4}[2x_3^2 - 4hx_3 + 2h^2] + \frac{2M_3S'_{2323}}{\pi R^4}[x_2x_3 - hx_2] + S'_{1133}\left[\frac{2M_2[x_1^2 - x_2^2]}{\pi R^4} - \frac{4x_2x_1M_1}{\pi R^4} - \frac{x_1F_3}{\pi R^2}\right] \\
u_2^0 &= \frac{M_1S'_{3333}}{\pi R^4}[2x_3^2 - 4hx_3 + 2h^2] - \frac{2M_3S'_{2332}}{\pi R^4}[x_1x_3 + hx_1] + S'_{1133}\left[\frac{4x_2x_1M_2}{\pi R^4} + \frac{2M_1[x_1^2 - x_2^2]}{\pi R^4} - \frac{x_2F_3}{\pi R^2}\right] \\
u_3^0 &= S'_{3333}\left[\frac{4x_1M_2}{\pi R^4} - \frac{4x_2M_1}{\pi R^4} - \frac{F_3}{\pi R^2}\right]x_3 + S'_{3333}\left[\frac{4x_2}{\pi R^4}M_1 - \frac{4x_1}{\pi R^4}M_2 + \frac{F_3}{\pi R^2}\right]h
\end{aligned}
\tag{7.6}
$$

To compare (7.6) to the displacement of the neutral fiber as given by elementary beam theory set $x_1 = x_2 = 0$, then

$$\begin{aligned} u_1^0(0,0,x_3) &= -\frac{M_2 S'_{3333}}{\pi R^4}[2x_3^2 - 4hx_3 + 2h^2] \\ u_2^0(0,0,x_3) &= \frac{M_1 S'_{3333}}{\pi R^4}[2x_3^2 - 4hx_3 + 2h^2] \\ u_3^0(0,0,x_3) &= \frac{F_3 S'_{3333}}{\pi R^2}[h - x_3] \end{aligned} \tag{7.7}$$

8. Implications of using the relaxed Saint-Venant's problem

In deriving (3.3) Iesan (1987) proved if $\mathbf{u}$ is a solution to the class of problems P_1 then $\mathbf{u}_{,3}$ is also a solution. The displacement $\mathbf{u}_{,3}$ can be represented by the rigid body motion

$$\mathbf{u}_{,3} = \boldsymbol{\alpha} + \boldsymbol{\beta} \times \mathbf{x} \tag{8.1}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are constant vectors.

In forming the problem in Fig. 3 as a relaxed Saint-Venant's problem, the fixed condition on Σ_2 is replaced by a stress field that is in equilibrium with the resultant loads applied to Σ_1. The removal of the fixed condition on Σ_2 allows the nontrivial rigid body motion (8.1). If the fixed condition on Σ_2 were maintained then there would only be the trivial rigid body motion

$$\mathbf{u}_{,3} = \boldsymbol{\alpha} + \boldsymbol{\beta} \times \mathbf{x} = \mathbf{0} \tag{8.2}$$

The infinitesimal strains are a function of the first partial derivatives of the displacements. Integrating the rigid body motion (8.1) with respect to x_3 results in the displacement field $\mathbf{u}$ being linear in x_1 and x_2. In addition, the displacement field $\mathbf{u}$ can be at most a function of the second power of x_3, where the terms containing x_3^2 are independent of x_1 and x_2. Therefore, the strains can at most be linear functions of x_i, which can be shown by substituting (5.4), (6.2), and (6.9) into (3.6). If the strains are linear in x_i, then by (2.2b) and (2.6) the stresses must be linear in x_i, which is shown by (5.5).

Thus in posing the problem in Fig. 3 as a relaxed Saint-Venant's problem, Iesan (1987) was effectively assuming the strains, and therefore the stresses, to be linear in x_i. The result of assuming the strains are linear in x_i is that plane sections remain plain for the material considered in this paper. Assuming plane sections remain plane is one of the fundamental assumptions of elementary beam theory.

9. Conclusions

The displacements $\mathbf{u}^0$ given by (7.6) include displacement components in the plane of Σ, which are not considered by elementary beam theory. The three-dimensional displacement equations can be used to measure the compliance coefficients in cylindrical coordinates using full size specimens.

The relaxed Saint-Venant's problem, for the material considered in this paper, results in plane sections remaining plane. Since plane sections remain plane the displacement equations for the neutral fiber (7.7) and the stress equations (5.5) are the same as those given by elementary beam theory. It was found for the cylindrical section of a tree considered in this paper that $S_{\alpha\beta} = 0$. Therefore, by Eq. (1.3) it is appropriate to use elementary beam theory to estimate the three-dimensional stresses for the problem considered in this paper.

Appendix A. Transformation equations

The transformation equations taking the elasticity coefficients in cylindrical coordinates (C'_{ijkl}) to Cartesian coordinates (C_{ijkl})

$$
\begin{aligned}
C_{1111} &= C_\theta^4 C'_{1111} + 2C_\theta^2 S_\theta^2 C'_{1122} + 4C_\theta^2 S_\theta^2 C'_{1212} + S_\theta^4 C'_{2222} \\
C_{2222} &= S_\theta^4 C'_{1111} + 2C_\theta^2 S_\theta^2 C'_{1122} + 4C_\theta^2 S_\theta^2 C'_{1212} + C_\theta^4 C'_{2222} \\
C_{3333} &= C'_{3333} \\
C_{2323} &= S_\theta^2 C'_{1313} + C_\theta^2 C'_{2323} \\
C_{1313} &= C_\theta^2 C'_{1313} + S_\theta^2 C'_{2323} \\
C_{1212} &= C_\theta^2 S_\theta^2 \left[C'_{1111} - 2C'_{1122} + C'_{2222} - 2C'_{1212}\right] + \left[C_\theta^4 + S_\theta^4\right] C'_{1212} \\
C_{1122} &= C_\theta^2 S_\theta^2 C'_{1111} + C_\theta^4 C'_{1122} - 4C_\theta^2 S_\theta^2 C'_{1212} + S_\theta^4 C'_{2211} + C_\theta^2 S_\theta^2 C'_{2222} \\
C_{1133} &= C_\theta^2 C'_{1133} + S_\theta^2 C'_{2233} \\
C_{1123} &= 0 \\
C_{1113} &= 0 \\
C_{1112} &= -C_\theta S_\theta \left[C_\theta^2 C'_{1111} - C_\theta^2 C'_{1122} - 2C_\theta^2 C'_{1212} + 2S_\theta^2 C'_{1212} + S_\theta^2 C'_{1122} - S_\theta^2 C'_{2222}\right] \\
C_{2233} &= S_\theta^2 C'_{1133} + C_\theta^2 C'_{2233} \\
C_{2223} &= 0 \\
C_{2213} &= 0 \\
C_{2212} &= -C_\theta S_\theta \left[S_\theta^2 C'_{1111} - S_\theta^2 C'_{1122} - 2S_\theta^2 C'_{1212} + 2C_\theta^2 C'_{1212} + C_\theta^2 C'_{1122} - C_\theta^2 C'_{2222}\right] \\
C_{3323} &= 0 \\
C_{3313} &= 0 \\
C_{3312} &= -C_\theta S_\theta \left[C'_{3311} - C'_{3322}\right] \\
C_{2313} &= -C_\theta S_\theta \left[C'_{1313} - C'_{2323}\right] \\
C_{2312} &= 0 \\
C_{1312} &= 0
\end{aligned}
\tag{A.1}
$$

For the transformation equations taking the compliance coefficients in cylindrical coordinates (S'_{ijkl}) to Cartesian coordinates (S_{ijkl}), replace C'_{ijkl} with S'_{ijkl} and C_{ijkl} with S_{ijkl} in Eq. (A.1).

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